

SETS OF INDEPENDENT POSTULATES FOR THE ARITHMETIC MEAN, THE GEOMETRIC MEAN, THE HARMONIC MEAN, AND THE ROOT-MEAN-SQUARE*

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INTRODUCTION

The four types of means, or averages, considered in this paper are the following: the arithmetic mean (A); the geometric mean (G); the harmonic mean (H); and the root-mean-square (S); the familiar definitions being as follows:

$$\begin{aligned} A &= \frac{1}{n} (x_1 + x_2 + \cdots + x_n), \\ G &= (x_1 x_2 \cdots x_n)^{1/n}, \\ H &= \frac{1}{\frac{1}{n} \left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} \right)}, \\ S &= \left(\frac{1}{n} (x_1^2 + x_2^2 + \cdots + x_n^2) \right)^{1/2}. \end{aligned}$$

The first three are the classical means, known to the Greeks, while the root-mean-square is of more modern origin. Of the four types, perhaps the most important are the *arithmetic mean* and the *root-mean-square*. Both of these averages are in constant use in mechanics (as in the definitions of center of gravity and radius of gyration), and in the modern theory of statistics (as in the theory of probability, the theory of least squares, and the definition of standard deviation). The *geometric mean* is important chiefly in the construction of index numbers. The *harmonic mean* is little used, except in special investigations.

Each of the four quantities A , G , H , S is a particular type of the general function of n variables: $f(x_1, x_2, \cdots, x_n)$; and the purpose of the present

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paper is to exhibit a number of *sets of independent postulates* by which each of these four types may be distinguished.

Unless otherwise stated, the variables $x_1, x_2, \dots, x_n$ are supposed to be positive real numbers.

The only similar set of postulates for any of these means, as far as is known to the present writer, is a set of postulates for the arithmetic mean given by R. Schimmack* in 1909; the "complete independence" of Schimmack's postulates was established by R. D. Beetle† in 1915.

PROPERTIES COMMON TO ALL FOUR MEANS

It will be convenient to begin by stating the following general postulates, I-V, which are satisfied by all four of the types of mean here considered. Each of these postulates is a condition imposed upon the as yet undetermined function $f(x_1, x_2, \dots, x_n)$ of the n positive real numbers $x_1, x_2, \dots, x_n$. Various selections from these postulates will be made below.

$$\text{I. } f(x_1, x_2, \dots, x_i, x_j, \dots, x_n) = f(x_1, x_2, \dots, x_j, x_i, \dots, x_n).$$

That is, the function " f " is independent of the order in which the n quantities $x_1, x_2, \dots, x_n$ are taken.

$$\text{II. } f(x_1, x_2, x_3, \dots, x_n) = f(m, m, x_3, \dots, x_n) \text{ where } m = f(x_1, x_2).$$

That is, in computing the " f " of n quantities, we may replace the first pair, x_1, x_2 , by the " f " of that pair, entered twice.

$$\text{III. } f(kx_1, kx_2, \dots, kx_n) = kf(x_1, x_2, \dots, x_n) \quad (k \text{ positive}).$$

That is, multiplying each of the n quantities by a positive factor k has the effect of multiplying the " f " of those quantities by the same factor k . In other words, the function " f " is independent of the scale in which the quantities $x_1, x_2, \dots, x_n$ are measured.

$$\text{IV. } f(a, a, \dots, a) = a.$$

That is, if the n quantities are all equal, then their " f " is equal to their common value.

$$\text{V. } f(x_1, x_2, \dots, x_n) \text{ is positive when all the } x\text{'s are positive.}$$

The postulates III and IV may sometimes be replaced, as we shall see, by the following weaker forms:

*R. Schimmack, *Der Satz vom arithmetischen Mittel in axiomatischer Begründung*, Mathematische Annalen, vol. 68 (1909), pp. 125-132, and p. 304.

†R. D. Beetle, *On the complete independence of Schimmack's postulates for the arithmetic mean*, Mathematische Annalen, vol. 76 (1915), pp. 444-446.

$$\text{III}'. \quad f(kx_1, kx_2) = kf(x_1, x_2) \quad (k \text{ positive}).$$

$$\text{IV}'. \quad f(1, 1, \dots, 1) = 1.$$

It may be noted that Postulates I, II, III, and III' have the form of "functional equations," in which the function " f " appears on both sides of the equality sign; while Postulates IV, V, and IV' are analogous to the "boundary conditions" of a problem in differential equations, since they tell us something about the actual value of the function in certain cases.

We now turn to properties which are peculiar to the several types of mean. (It will be observed that in each of the following four groups, the first postulate is a "boundary condition," while the other three postulates are "functional equations.") Various selections from these postulates will be made below.

POSTULATES PECULIAR TO THE ARITHMETIC MEAN (POSITIVE QUANTITIES*)

$$A1. \quad f(a, b) = \frac{1}{2}(a + b).$$

$$A2. \quad f(1 - a, 1 - b) = 1 - f(a, b) \quad (a < 1, b < 1).$$

$$A3. \quad f(1 - x_1, 1 - x_2, \dots, 1 - x_n) = 1 - f(x_1, x_2, \dots, x_n) \quad (x_i < 1).$$

$$A4. \quad f(A - x_1, A - x_2, \dots, A - x_n) = A - f(x_1, x_2, \dots, x_n) \text{ for all values of } A \text{ for which } A - x_i > 0.$$

POSTULATES PECULIAR TO THE GEOMETRIC MEAN

$$G1. \quad f(a, b) = (ab)^{1/2}.$$

$$G2. \quad f\left(\frac{1}{a}, \frac{1}{b}\right) = \frac{1}{f(a, b)}.$$

$$G3. \quad f\left(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_n}\right) = \frac{1}{f(x_1, x_2, \dots, x_n)}.$$

$$G4. \quad f\left(\frac{A}{x_1}, \frac{A}{x_2}, \dots, \frac{A}{x_n}\right) = \frac{A}{f(x_1, x_2, \dots, x_n)},$$

where A is positive. Here $(ab)^{1/2}$ means $+(ab)^{1/2}$ not $-(ab)^{1/2}$.

POSTULATES PECULIAR TO THE HARMONIC MEAN

$$H1. \quad f(a, b) = \frac{2ab}{a + b}.$$

*For further postulates A8, A8', A9, intended for use in the domain of all real or all complex quantities, see Appendix I and Appendix II.

$$H2. \quad \left(\frac{a}{a-1}, \frac{b}{b-1} \right) = \frac{f(a, b)}{f(a, b) - 1}.$$

$$H3. \quad f\left(\frac{x_1}{x_1-1}, \frac{x_2}{x_2-1}, \dots, \frac{x_n}{x_n-1}\right) = \frac{f(x_1, x_2, \dots, x_n)}{f(x_1, x_2, \dots, x_n) - 1} \quad (x_i > 1).$$

$$H4. \quad f\left(\frac{x_1}{Ax_1-1}, \frac{x_2}{Ax_2-1}, \dots, \frac{x_n}{Ax_n-1}\right) = \frac{f(x_1, x_2, \dots, x_n)}{Af(x_1, x_2, \dots, x_n) - 1}$$

for all values of A for which $Ax_i - 1 > 0$.

POSTULATES PECULIAR TO THE ROOT-MEAN-SQUARE

$$S1. \quad f(a, b) = \{(x_1^2 + x_2^2)/2\}^{1/2}.$$

$$S2. \quad f\{(1 - a^2)^{1/2}, (1 - b^2)^{1/2}\} = \{1 - [f(a, b)]^2\}^{1/2} \quad (a < 1, b < 1).$$

$$S3. \quad f\{(1 - x_1^2)^{1/2}, (1 - x_2^2)^{1/2}, \dots, (1 - x_n^2)^{1/2}\} \\ = \{1 - [f(x_1, x_2, \dots, x_n)]^2\}^{1/2} \quad (x_i < 1).$$

$$S4. \quad f\{f(A - x_1^2)^{1/2}, (A - x_2^2)^{1/2}, \dots, (A - x_n^2)^{1/2}\} \\ = \{A - [f(x_1, x_2, \dots, x_n)]^2\}^{1/2},$$

for all values of A for which $A - x_i^2 > 0$. Here again, $x^{1/2}$ means $+x^{1/2}$ not $-x^{1/2}$.

SETS OF INDEPENDENT POSTULATES FOR EACH TYPE OF MEAN.

Among these general and special properties there are, of course, many redundancies. The purpose of the present paper is to select, for each type of mean, sets of *independent* postulates; that is, sets of postulates which, while sufficient to determine uniquely the type of mean in question, shall at the same time be free from all redundancies. Such sets can be selected in a variety of ways, as in the following tables. These tables give the postulates belonging to each set, and also the list of examples which will later be used to prove the independence of the postulates in that set.

It will be observed that in each group the fifth and sixth sets are obtained from the second set by replacing either III or IV by the weaker form III' or IV'; while the seventh set is obtained from the first by replacing IV by IV' and then adding III. No further replacements of III or IV by III' or IV' are possible, as we shall see by Examples *A* III 7, *G* III 7, *H* III 7, and *S* III 7.

Thus, a set of postulates comprising *A*1, *A*2, *A*3, I, II, III', IV', V would not be sufficient to determine the arithmetic mean; *G*1, *G*2, *G*3, I, II, III', IV', V would not be sufficient for the geometric mean; nor *H*1, *H*2, *H*3, I, II, III', IV', V for the harmonic mean; nor *S*1, *S*2, *S*3, I, II, III', IV', V for the root-mean-square.

For the Arithmetic Mean (positive quantities*)

Set	Postulates	Examples Used
A1	A1 I II IV	0 A I A II A IV
A2	A2 I II III IV	0 A I A II A III A IV
A3	A3 I II III	0 A I A II A III
A4	A4 I II	0 A I A II
A5	A2 I II III' IV	0 A I A II A III A IV
A6	A2 I II III IV'	0 A I A II A III A IV
A7	A1 I II III IV'	0 A I A II A III 7 A IV

For the Geometric Mean

Set	Postulates	Examples Used
G1	G1 I II IV	0 G I G II G IV
G2	G2 I II III IV V	0 G I G II G III G IV G V
G3	G3 I II III V	0 G I G II G III G V
G4	G4 I II V	0 G I G II G IV G V
G5	G2 I II III' IV V	0 G I G II G III G IV G V
G6	G2 I II III IV' V	0 G I G II G III G IV G V
G7	G1 I II III IV'	0 G I G II G III 7 G IV

For the Harmonic Mean

Set	Postulates	Examples Used
H1	H1 I II IV	0 H I H II H IV
H2	H2 I II III IV V	0 H I H II H III H IV H V
H3	H3 I II III V	0 H I H II H III H V
H4	H4 I II V	0 H I H II H V
H5	H2 I II III' IV V	0 H I H II H III H IV H V
H6	H2 I II III IV' V	0 H I H II H III H IV H V
H7	H1 I II III IV'	0 H I H II H III 7 H IV

*For further Sets A6'', A7'', A8, A9, intended for use in the real or complex domain, see Appendices I and II.

For the Root-Mean-Square

Set	Postulates					Examples Used				
S1	S1	I	II		IV	0	SI	S II		S IV
S2	S2	I	II	III	IV	0	SI	S II	S III	S IV
S3	S3	I	II	III		0	SI	S II	S III	
S4	S4	I	II			0	SI	S II		
S5	S2	I	II	III'	IV	0	SI	S II	S III	S IV
S6	S2	I	II	III	IV'	0	SI	S II	S III	S IV
S7	S1	I	II	III	IV'	0	SI	S II	S III	S IV

EXAMPLES USED IN PROOFS OF INDEPENDENCE

To establish the independence of the postulates of each set, we exhibit, in the usual way, a list of examples of functions $f(x_1, x_2, \dots, x_n)$ which satisfy some but not all of the postulates. In the following table the postulates satisfied by each example are stated explicitly, opposite the number of that example. (A dash, —, indicates that the postulate is not satisfied.) The list of examples is given immediately below the table.*

As an illustration of the use of these examples, consider Postulate III in Set G3. Example G III satisfies Postulates G3, I, II, and V, but fails on Postulate III. Hence III is not a consequence of G3, I, II, V; that is, Postulate III is not a redundancy in Set G3. Similarly for each of the other postulates in this set and in each of the other sets.

$$\text{Ex. 0.} \quad f() = \{(x_1^3 + x_2^3 + \dots + x_n^3)/n\}^{1/3}.$$

This example satisfies all the general postulates I–V, but none of the special postulates A1–A4, G1–G4, H1–H4, S1–S4. To see that II is satisfied, note that

$$m = f(x_1, x_2) = \{(x_1^3 + x_2^3)/2\}^{1/3},$$

so that $m^3 + m^3 = x_1^3 + x_2^3$.

$$\text{Ex. AI.} \quad f() = \frac{x_1 + x_2 + 3x_3 + 4x_4 + \dots + nx_n}{1 + 1 + 3 + 4 + \dots + n}.$$

*For further Examples: 0', AII', A III'', A IV', A IV''; AII'', A IX, and AX, for the arithmetic mean in the real or complex domain, see Appendices I and II.

	A1 A2 A3 A4	I II III IV	III' IV'	V
Ex. 0	— — — —	I II III IV	III' IV'	V
Ex. A I	A1 A2 A3 A4	— II III IV	III' IV'	V
Ex. A II	A1 A2 A3 A4	I — III IV	III' IV'	V
Ex. A III	— A2 A3 —	I II — IV	— IV'	V
Ex. A IV	A1 A2 — —	I II III —	III' —	V
Ex. A III 7	A1 A2 A3 —	I II — —	III' IV'	V

	G1 G2 G3 G4	I II III IV	III' IV'	V
Ex. 0	— — — —	I II III IV	III' IV'	V
Ex. G I	G1 G2 G3 G4	— II III IV	III' IV'	V
Ex. G II	G1 G2 G3 G4	I — III IV	III' IV'	V
Ex. G III	— G2 G3 —	I II — IV	— IV'	V
Ex. G IV	G1 G2 — —	I II III —	III' —	V
Ex. G III 7	G1 G2 G3 —	I II — —	III' IV'	V
Ex. G V	— G2 G3 G4	I II III IV	III' IV'	—

	H1 H2 H3 H4	I II III IV	III' IV'	V
Ex. 0	— — — —	I II III IV	III' IV'	V
Ex. H I	H1 H2 H3 H4	— II III IV	III' IV'	V
Ex. H II	H1 H2 H3 H4	I — III IV	III' IV'	V
Ex. H III	— H2 H3 —	I II — IV	— IV'	V
Ex. H IV	H1 H2 — —	I II III —	III' —	V
Ex. H III 7	H1 H2 H3 —	I II — —	III' IV'	V
Ex. H V	— H2 H3 H4	I II III IV	III' IV'	—

	S1 S2 S3 S4	I II III IV	III' IV'	V
Ex. 0	— — — —	I II III IV	III' IV'	V
Ex. S I	S1 S2 S3 S4	— II III IV	III' IV'	V
Ex. S II	S1 S2 S3 S4	I — III IV	III' IV'	V
Ex. S III	— S2 S3 —	I II — IV	— IV'	V
Ex. S IV	S1 S2 — —	I II III —	III' —	V
Ex. S III 7	S1 S2 S3 —	I II — —	III' IV'	V

Ex. A II. $f()$ = the "median" of the quantities $x_1, x_2, \dots, x_n$, if n is odd, or the arithmetic mean of the "median-pair" if n is even.* To see that II fails note that $f(5, 3, 7) = 5$, while $f(4, 4, 7) = 4$.

Ex. A III. $f(x, x, \dots, x) = x$ when all the x 's are equal; otherwise $f() = \frac{1}{2}$. This example fails on III and III', and on A1 and A4; but it satisfies A2 and A3.

Ex. A IV. $f() = (x_1 + x_2 + \dots + x_n)/2$.

This example fails on IV and IV', and on A3 and A4; but it satisfies A1 and A2.

Ex. A III 7. $f(x_1, x_2) = (x_1 + x_2)/2$; but when $n > 2$, $f() = 1$ or $\frac{1}{2}$, according as $(x_1 + x_2 + \dots + x_n)/n$ is equal to 1 or not equal to 1. This example is used only in Set A7, to prove the independence of Postulate III in that set. It fails on III and IV, but satisfies III' and IV'. To see that it satisfies A2 and A3, note that if all the x 's are less than 1, the value of $(x_1 + x_2 + \dots + x_n)/n$ must be less than 1, and hence $f() = \frac{1}{2}$.

Ex. G I. $f() = (x_1 x_2 x_3 x_4 \dots x_n)^{1/p}$, where $p = 1 + 1 + 3 + 4 + \dots + n$.

Ex. G II. $f()$ = the median of the n quantities if n is odd, or the geometric mean of the median-pair if n is even. To see that II fails, note that $f(4, 9, 5) = 5$, while $f(6, 6, 5) = 6$.

Ex. G III. $f(x, x, \dots, x) = x$ when all the x 's are equal; otherwise $f() = 1$.

Ex. G IV. $f() = \{(x_1 x_2 \dots x_n)/(n-1)\}^{1/n}$. To see that II holds, note that $m = f(x_1, x_2) = (x_1 x_2)^{1/2}$, so that $mm = x_1 x_2$.

Ex. G III 7. $f(x_1, x_2) = (x_1 x_2)^{1/2}$; but when $n > 2$, $f() = 1$. This example is used only in Set G7, to prove the independence of Postulate III in that set.

Ex. G V. $f(x, x, \dots, x) = x$ when all the x 's are equal; otherwise $f() = -|x_1 x_2 \dots x_n|^{1/n}$, where the expression of which the n th root is taken is the absolute value of the product of the x 's without regard to sign.

$$\text{Ex. H I.} \quad f() = \frac{1 + 1 + 3 + 4 + \dots + n}{\frac{1}{x_1} + \frac{1}{x_2} + \frac{3}{x_3} + \frac{4}{x_4} + \dots + \frac{n}{x_n}}.$$

Here when $n = 2$,

$$f(x_1, x_2) = \frac{2x_1 x_2}{x_1 + x_2}.$$

Ex. H II. $f()$ = the median of the n quantities if n is odd, or the harmonic mean of the median-pair if n is even. To see that II fails, note that $f(3, 6, 2) = 3$, while $f(4, 4, 2) = 4$.

*Here the *median* (or *median-pair*) of n positive quantities is defined by arranging the n quantities in a series in order of magnitude, and picking out the middle item (or mid-pair) in this series.

Ex. *H* III. $f(x, x, \dots, x) = x$ if all the x 's are equal; otherwise $f() = 2$.

Ex. *H* IV.
$$f() = 2 / \left(\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} \right).$$

Ex. *H* III 7. $f(x_1, x_2) = 2x_1x_2/(x_1+x_2)$; but when $n > 2$, $f() = 1$ or 2 , according as $(1/n)[(1/x_1) + (1/x_2) + \dots + (1/x_n)]$ is equal to 1 or not equal to 1 . This example is used only in Set *H7*, to prove the independence of Postulate III in that set.

Ex. *H* V. $f(x, x, \dots, x) = x$ when all the x 's are equal; otherwise $f() = 0$.

Ex. *S* I.
$$f() = \left(\frac{x_1^2 + x_2^2 + 3x_3^2 + 4x_4^2 + \dots + nx_n^2}{1 + 1 + 3 + 4 + \dots + n} \right)^{1/2}.$$

To see that this example satisfies II, note that $m = f(x_1, x_2) = \{(x_1^2 + x_2^2)/2\}^{1/2}$, so that $m^2 + m^2 = x_1^2 + x_2^2$.

Ex. *S* II. $f()$ = the median of the n quantities if n is odd, or the root-mean-square of the median-pair if n is even. To see that II fails, note that $f(6, 8, 4) = 6$, while $f(50^{1/2}, 50^{1/2}, 4) = 50^{1/2}$.

Ex. *S* III. $f(x, x, \dots, x) = x$ when all the x 's are equal; otherwise $f() = (1/2)^{1/2}$.

Ex. *S* IV. $f() = \{(x_1^2 + x_2^2 + \dots + x_n^2)/2\}^{1/2}$.

Ex. *S* III 7. $f(x_1, x_2) = \{(x_1^2 + x_2^2)/2\}^{1/2}$; but when $n > 2$, $f() = 1$ or $(1/2)^{1/2}$, according as $(x_1^2 + x_2^2 + \dots + x_n^2)/n$ is equal to 1 or not equal to 1 . This example is used only in Set *S7*, to prove the independence of Postulate III in that set.

PROOFS OF THEOREMS

In the following paragraphs, we give the proof that each of the foregoing sets of postulates is sufficient to define the type of mean in question.

THEOREM A (A). *Proof of A from A1, I, II, and IV.*

Let q be any positive quantity which is less than $(1/n)$ th of the smallest of the x 's. Then by II and A1,

$$f(x_1, x_2, x_3, \dots, x_n) = f(q, [x_1 + x_2 - q], x_3, \dots, x_n),$$

since each side equals $f(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(x_1 + x_2), x_3, \dots, x_n)$, and all the arguments are positive.

By successive applications of this result, in view of I, we have

$$\begin{aligned} f(x_1, x_2, \dots, x_n) &= f(q, q, [x_1 + x_2 + x_3 - 2q], x_4, \dots, x_n) \\ &= f(q, q, q, [x_1 + x_2 + x_3 + x_4 - 3q], x_5, \dots, x_n) \\ &= f(q, q, \dots, [x_1 + x_2 + \dots + x_n - (n-1)q]). \end{aligned}$$

Now take $a = (x_1 + x_2 + \cdots + x_n)/n$. Then, putting each x equal to a , $f(a, a, \cdots, a) = f(q, q, \cdots, [na - (n-1)q]) = a$, by Postulate IV. But $x_1 + x_2 + \cdots + x_n = na$, so that

$$f(x_1, x_2, \cdots, x_n) = f(q, q, \cdots, [na - (n-1)q]).$$

Hence

$$f(x_1, x_2, \cdots, x_n) = (x_1 + x_2 + \cdots + x_n)/n.$$

Hence any function f which satisfies the postulates of Set A1 must be identical with the arithmetic mean, A .

THEOREM A (B). *Proof of A1 from A2, I, and III or III'.*

$$\begin{aligned} f(a, b) &= (a+b)f\left(\frac{a}{a+b}, \frac{b}{a+b}\right), \text{ by III or III' ;} \\ &= (a+b)f\left(1 - \frac{b}{a+b}, 1 - \frac{a}{a+b}\right) \\ &= (a+b)\left[1 - f\left(\frac{b}{a+b}, \frac{a}{a+b}\right)\right], \text{ by A2 ;} \\ &= (a+b) - f(b, a), \text{ by III or III' ;} \\ &= (a+b) - f(a, b), \text{ by I.} \end{aligned}$$

Hence $f(a, b) = (a+b)/2$, which is A1.

This proof shows that any function which satisfies the postulates of Set A2 or Set A5 will also satisfy the postulates of Set A1, and hence be identical with the arithmetic mean.

THEOREM A (C). *Proof of IV from A3, III.*

From A3, putting $x_1 = \frac{1}{2}, x_2 = \frac{1}{2}, \cdots, x_n = \frac{1}{2}$, we have

$$f\left(\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}\right) = 1 - f\left(\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}\right),$$

whence $f\left(\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}\right) = \frac{1}{2}$. Hence by III, $f(a, a, \cdots, a) = a$, which is IV.

This proof shows (since Postulate A2 follows at once from Postulate A3) that any function which satisfies Set A3 will also satisfy Set A2.

THEOREM A (D). *Proof of A1 and IV from A4, I.*

From A4, putting $A = a+b, x_1 = a, x_2 = b$, and $n=2$, we have

$$f(a+b-a, a+b-b) = a+b-f(a, b);$$

whence by I, $f(a, b) = a+b-f(a, b)$. Therefore $f(a, b) = (a+b)/2$, which is A1.

From $G3$, putting $x_1=1, x_2=1, \dots, x_n=1$, we have $f(1, 1, \dots, 1) = 1/f(1, 1, \dots, 1)$, whence $f(1, 1, \dots, 1) = 1$ or -1 . Hence by V , $f(1, 1, \dots, 1) = 1$. Then by III , $f(a, a, \dots, a) = a$, which is IV .

This proof shows (since $G2$ follows at once from $G3$) that Set $G3$ reduces to Set $G2$, and hence determines the geometric mean.

THEOREM G (D). *Proof of $G1$ and IV from $G4$, I, V.*

From $G4$, putting $a=ab, x_1=a, x_2=b$, and $n=2$, we have

$$f\left(\frac{ab}{a}, \frac{ab}{b}\right) = \frac{ab}{f(a, b)}.$$

Hence by I ,

$$f(c, b) = \frac{ab}{f(a, b)}.$$

Therefore $f(a, b) = (ab)^{1/2}$ or $-(ab)^{1/2}$. But the negative value is excluded, by V . Hence $f(a, b) = (ab)^{1/2}$, which is $G1$.

Again, from $G4$, putting x equal to a , and $A=a^2$, we have

$$f(a, a, \dots, a) = \frac{a^2}{f(a, a, \dots, a)},$$

whence, by V , $f(a, a, \dots, a) = a$, which is IV .

This proof shows that Set $G4$ reduces to Set $G1$, and hence determines the geometric mean.

THEOREM G (E). Since IV follows at once from III and IV' , we see that Set $G6$ reduces to Set $G2$, and that Set $G7$ reduces to Set $G1$; hence Set $G6$ and Set $G7$ determine the geometric mean.

THEOREM H (A). *Proof of H from $H1$, I, II, IV.*

Let q be any positive quantity which is greater than n times the largest of the x 's. Then by II and $H1$,

$$f(x_1, x_2, \dots, x_n) = f\left(q, \frac{1}{\frac{1}{x_1} + \frac{1}{x_2} - \frac{1}{q}}, x_3, \dots, x_n\right),$$

since each side is equal to

$$f\left(\frac{2x_1x_2}{x_1 + x_2}, \frac{2x_1x_2}{x_1 + x_2}, x_3, \dots, x_n\right),$$

and all the arguments are positive. By successive applications of this result, using I, we have

$$\begin{aligned} f(x_1, x_2, \dots, x_n) &= f\left(q, q, \frac{1}{\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} - \frac{2}{q}}, x_4, \dots, x_n\right) \\ &\quad \cdot \cdot \cdot \cdot \cdot \cdot \\ &= f\left(q, q, \dots, \frac{1}{\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} - \frac{n-1}{q}}\right). \end{aligned}$$

Now take

$$a = \left(\frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n}} \right).$$

Then putting each x equal to a ,

$$f(a, a, \dots, a) = \left(q, q, \dots, \frac{1}{\frac{n}{a} - \frac{n-1}{q}} \right) = a,$$

by Postulate IV. But

$$\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} = \frac{n}{a},$$

so that

$$f(x_1, x_2, \dots, x_n) = f\left(q, q, \dots, \frac{1}{\frac{n}{a} - \frac{n-1}{q}}\right) = a.$$

Hence

$$f(x_1, x_2, \dots, x_n) = \frac{1}{\frac{1}{n} \left(\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} \right)}.$$

This proof shows that Set $H1$ determines the harmonic mean.

THEOREM H (B). *Proof of H1 from H2, I, III or III', V.*

$$\begin{aligned}
 f(a, b) &= \frac{ab}{a+b} f\left(\frac{a+b}{b}, \frac{a+b}{a}\right), \quad \text{by III or III'}; \\
 &= \frac{ab}{a+b} f\left[\frac{\frac{a+b}{a}}{\frac{a+b}{a}-1}, \frac{\frac{a+b}{b}}{\frac{a+b}{b}-1}\right] \\
 &= \frac{ab}{a+b} \frac{f\left(\frac{a+b}{a}, \frac{a+b}{b}\right)}{f\left(\frac{a+b}{a}, \frac{a+b}{b}\right)-1}, \quad \text{by H2.}
 \end{aligned}$$

Now by V, $f(a, b)$ is not zero. Hence

$$\begin{aligned}
 \frac{1}{f(a, b)} &= \frac{1}{\frac{ab}{a+b}} - \frac{1}{\frac{ab}{a+b} f\left(\frac{a+b}{a}, \frac{a+b}{b}\right)} \\
 &= \frac{a+b}{ab} - \frac{1}{f(b, a)}, \quad \text{by III or III'}; \\
 &= \frac{a+b}{ab} - \frac{1}{f(a, b)}, \quad \text{by I.}
 \end{aligned}$$

Hence

$$f(a, b) = \frac{2ab}{a+b},$$

which is H1.

This proof shows that Set H2 and Set H5 reduce to Set H1, and hence determine the harmonic mean.

THEOREM H (C). *Proof of IV from H3, III, V.*

From H3, putting $x_1=2, x_2=2, \dots, x_n=2$, we have

$$f(2, 2, \dots, 2) = \frac{f(2, 2, \dots, 2)}{f(2, 2, \dots, 2) - 1}.$$

Hence $f(2, 2, \dots, 2)[f(2, 2, \dots, 2)-2]=0$. But by V, $f(2, 2, \dots, 2)$ is not zero. Hence $f(2, 2, \dots, 2)=2$. Hence by III, $f(a, a, \dots, a)=a$, which is IV.

This proof shows (since *H2* follows at once from *H3*) that Set *H3* reduces to Set *H2*, and hence determines the harmonic mean.

THEOREM H (D). *Proof of H1 and IV from H4.*

From *H4*, putting $A = (a+b)/(ab)$, $x_1 = a$, $x_2 = a$, and $n=2$, we have

$$f(b, a) = \frac{(ab)f(a, b)}{(a+b)f(a, b) - ab},$$

whence by I,

$$f(a, b) = \frac{(ab)f(a, b)}{(a+b)f(a, b) - ab},$$

where by V, $f(a, b)$ is not zero; hence $f(a, b) = 2ab/(a+b)$, which is *H1*.

Again, from *H4*, putting each x equal to a , and $A = 2/a$, we have

$$f(a, a, \dots, a) = \frac{af(a, a, \dots, a)}{2f(a, a, \dots, a) - a},$$

where by V, $f(a, a, \dots, a)$ is not zero; hence $f(a, a, \dots, a) = a$, which is IV.

This proof shows that Set *H4* reduces to Set *H1*, and hence determines the harmonic mean.

THEOREM H (E). Since IV follows at once from III and IV', we see that Set *H6* and Set *H7* reduce to Set *H2* and Set *H1* respectively, and hence determine the harmonic mean.

THEOREM S (A). *Proof of S from S1, I, II, IV.*

Let q be any positive quantity which is smaller than $(1/n)$ th of the smallest of the x 's. Then by II and S1,

$$f(x_1, x_2, x_3, \dots, x_n) = f\{q, (x_1^2 + x_2^2 - q^2)^{1/2}, x_3, \dots, x_n\},$$

since each side is equal to $f\{[(x_1^2 + x_2^2)/2]^{1/2}, [(x_1^2 + x_2^2)/2]^{1/2}, x_3, \dots, x_n\}$, and all the arguments are positive. By successive applications of this result, in view of I, we have

$$\begin{aligned} f(x_1, x_2, \dots, x_n) &= f\{q, q, (x_1^2 + x_2^2 + x_3^2 - 2q^2)^{1/2}, x_4, \dots, x_n\} \\ &\quad \dots \dots \dots \\ &= f\{q, q, \dots, (x_1^2 + x_2^2 + \dots + x_n^2 - (n-1)q^2)^{1/2}\}. \end{aligned}$$

Now take $a = \{(x_1^2 + x_2^2 + \dots + x_n^2)/n\}^{1/2}$. Then putting each x equal to a , $f(a, a, \dots, a) = f\{q, q, \dots, (na^2 - (n-1)q^2)^{1/2}\} = a$, by IV.

But $x_1^2 + x_2^2 + \dots + x_n^2 = na^2$, so that

$$f(x_1, x_2, \dots, x_n) = f\{q, q, \dots, (na^2 - (n-1)q^2)^{1/2}\} = a.$$

Hence

$$f(x_1, x_2, \dots, x_n) = \left(\frac{x_1^2 + x_2^2 + \dots + x_n^2}{n} \right)^{1/2}.$$

This proof shows that Set S1 determines the root-mean-square.

THEOREM S (B). *Proof of S1 from S2, I, III or III'.*

$$\begin{aligned} f(a, b) &= (a^2 + b^2)^{1/2} f\left(\frac{a}{(a^2 + b^2)^{1/2}}, \frac{b}{(a^2 + b^2)^{1/2}}\right), \text{ by III or III' ;} \\ &= (a^2 + b^2)^{1/2} f\left[\left(1 - \frac{b^2}{a^2 + b^2}\right)^{1/2}, \left(1 - \frac{a^2}{a^2 + b^2}\right)^{1/2}\right] \\ &= (a^2 + b^2)^{1/2} \left(1 - \left[f\left(\frac{b}{(a^2 + b^2)^{1/2}}, \frac{a}{(a^2 + b^2)^{1/2}}\right)\right]^2\right)^{1/2}, \text{ by S2;} \\ &= \{(a^2 + b^2) - [f(b, a)]^2\}^{1/2}, \text{ by III or III' ;} \\ &= \{(a^2 + b^2) - [f(a, b)]^2\}^{1/2} \text{ by I.} \end{aligned}$$

Hence $2 [f(a, b)]^2 = a^2 + b^2$, so that

$$f(a, b) = \{(a^2 + b^2)/2\}^{1/2} \text{ or } - \{(a^2 + b^2)/2\}^{1/2}.$$

It remains to exclude the negative value (without using V). By S2, $f(a, b)$ is positive whenever $a < 1$ and $b < 1$; hence by III, $f(a, b)$ is positive for all positive values of a and b . Hence $f(a, b) = \{(a^2 + b^2)/2\}^{1/2}$.

This proof shows that Set S2 and Set S5 reduce to Set S1, and hence determine the root-mean-square.

THEOREM S (C). *Proof of IV from S3, III.*

From S3, putting $x_1 = \frac{1}{2}$, $x_2 = \frac{1}{2}$, $\dots$, $x_n = \frac{1}{2}$, we have

$$\begin{aligned} f\{(1/2)^{1/2}, (1/2)^{1/2}, \dots, (1/2)^{1/2}\} \\ = (1 - [f\{(1/2)^{1/2}, (1/2)^{1/2}, \dots, (1/2)^{1/2}\}]^2)^{1/2} \end{aligned}$$

which by definition of the square root sign, is not negative. Hence

$$f\{(1/2)^{1/2}, (1/2)^{1/2}, \dots, (1/2)^{1/2}\} = (1/2)^{1/2}.$$

Hence by III,

$$f(a, a, \dots, a) = a.$$

This proof shows (since S2 follows at once from S3) that Set S3 reduces to Set S2, and hence determines the root-mean-square.

THEOREM S (D). *Proof of S1 and IV from S4, I.*

From S4, putting $A = a^2 + b^2$, $x_1 = a$, $x_2 = b$, and $n = 2$, we have

$$f(b, a) = (a^2 + b^2 - [f(a, b)]^2)^{1/2}.$$

Hence, by I, $f(a, b) = (a^2 + b^2 - [f(a, b)]^2)^{1/2}$, which is not negative. Hence $f(a, b) = \{(a^2 + b^2)/2\}^{1/2}$.

Again, from $S4$, putting x equal to a , and $A = 2a^2$, we have

$$f(a, a, \dots, a) = \{2a^2 - [f(a, a, \dots, a)]^2\}^{1/2},$$

which is not negative; hence $f(a, a, \dots, a) = a$, which is Postulate IV.

This proof shows that Set $S4$ reduces to Set $S1$, and hence determines the root-mean-square.

THEOREM S (E). Since IV follows at once from III and IV', we see that Set $S6$ and Set $S7$ reduce to Set $S2$ and Set $S1$ respectively, and hence determine the root-mean-square.

APPENDIX I. THE ARITHMETIC MEAN IN THE DOMAIN OF ALL REAL NUMBERS

In all the preceding sets of postulates, the x 's in the function $f(x_1, x_2, \dots, x_n)$ have been assumed to be *positive real quantities*. In the case of the geometric mean, the harmonic mean, and the root-mean-square, this restriction is a customary one, in order to insure that the function shall always be finite and single-valued. In the case of the arithmetic mean, however, the restriction is not essential.

This appendix, therefore, is devoted to a consideration of sets of postulates for the arithmetic mean in the domain of all real numbers.

For this purpose, we introduce, in addition to the postulates I-V, III', IV', A1-A4, given above, the following postulates:

POSTULATE IV''. $f(-1, -1, \dots, -1) = -1$.

POSTULATE A8. $f(A + x_1, A + x_2, \dots, A + x_n) = A + f(x_1, x_2, \dots, x_n)$.

POSTULATE A8'. $f(-x_1, -x_2, \dots, -x_n) = -f(x_1, x_2, \dots, x_n)$.

The results obtained may be summarized as follows:

In the first place, Sets A1, A2, A3, A4, A5, given above for the case of positive reals, are valid just as they stand for the case of all reals. (The necessary modifications in the proofs are given below.)

In the second place, Sets A6 and A7 are not valid for the case of all reals (see Example A IV'' below); but they can be made so by the addition of Postulate IV''.

In the third place, Postulates I, II, A8, A8' form a set (due essentially to Schimmack*), which is valid for the case of all reals, but cannot be used in the case of positive reals; this we shall call Set A8.

*Instead of our Postulate II, Schimmack (loc. cit.) uses the following postulate:

$$f(x_1, x_2, \dots, x_{n-1}, x_n) = f(m, m, \dots, m, x_n), \text{ where } m = f(x_1, x_2, \dots, x_{n-1});$$

and instead of our Example AI, the following example:

$$f() = (x_1/2^{n-1}) + (x_2/2^{n-1}) + (x_3/2^{n-2}) + (x_4/2^{n-3}) + \dots + (x_{n-1}/2^2) + (x_n/2).$$

The complete list of sets of independent postulates for the arithmetic mean, for the case of all reals, is then as follows:

Set	Postulates						Examples Used					
A1	A1	I	II		IV		0	A I	A II'		A IV	
A2	A2	I	II	III	IV		0	A I	A II'	A III	A IV	
A3	A3	I	II	III			0	A I	A II'	A III		
A4	A4	I	II				0	A I	A II'			
A5	A2	I	II	III'	IV		0	A I	A II'	A III	A IV	
A6''	A2	I	II	III	IV'	IV''	0	A I	A II'	A III''	A IV'	A IV''
A7''	A1	I	II	III	IV'	IV''	0	A I	A II'	A III''	A IV'	A IV''
A8	A8	I	II	A8'			0	A I	A II'	0'		

The new examples in the independence proofs are the following:

Ex. A II'. The same as Ex. A II, for all reals, understanding "the order of magnitude" in the algebraic sense.

Ex. A III''. When $n=2, f(x_1, x_2) = \frac{1}{2}(x_1+x_2)$; when $n>2$,
 $f(x_1, x_2, \dots, x_n) = 1$ or -1
according as $(x_1+x_2+\dots+x_n)/n$ is equal to 1 or not equal to 1.

Ex. A IV'. When $n=2, f(x_1, x_2) = \frac{1}{2}(x_1+x_2)$; when $n>2$,
 $f() = -|(x_1+x_2+\dots+x_n)/n|$,
where the vertical bars mean "the absolute value of."

Ex. A IV''. When $n=2, f(x_1, x_2) = \frac{1}{2}(x_1+x_2)$; when $n>2$,
 $f() = |(x_1+x_2+\dots+x_n)/n|$.

Ex. 0'. $f()$ = the maximum (in the algebraic sense) of the n real quantities $x_1, x_2, \dots, x_n$.

The following table shows the properties of all the examples used in the case of the real domain.

	A1	A2	A3	A4	A8	A8'	I	II	III	IV	III'	IV'	IV''
Ex. 0	—	—	—	—	—	A8'	I	II	III	IV	III'	IV'	IV''
Ex. 0'	—	—	—	—	A8	—	I	II	III	IV	III'	IV'	IV''
Ex. A I	A1	A2	A3	A4	A8	A8'	—	II	III	IV	III'	IV'	IV''
Ex. A II'	A1	A2	A3	A4	A8	A8'	I	—	III	IV	III'	IV'	IV''
Ex. A III	—	A2	A3	—	—	—	I	II	—	IV	—	IV'	IV''
Ex. A IV	A1	A2	—	—	—	A8'	I	II	III	—	III'	—	—
Ex. A III''	A1	A2	—	—	—	—	I	II	—	—	III'	IV'	IV''
Ex. A IV'	A1	A2	—	—	—	—	I	II	III	—	III'	—	IV''
Ex. A IV''	A1	A2	—	—	—	—	I	II	III	—	III'	IV'	—

The proofs of the theorems, adapted to the domain of all reals, are as follows:

THEOREM A (A'). *Proof of A from A1, I, II, IV. (Real domain.)*

Since there is now no necessity for keeping the arguments positive, the proof of Theorem A (A) can be simplified by putting $q=0$. Hence Set A1 determines the arithmetic mean.

THEOREM A (B'). *Proof of A1 from A2, I, III or III'. (Real domain.)*

If a and b are positive, $f(a, b) = \frac{1}{2}(a+b)$, as in the proof of Theorem A (B).

If a and b are any real numbers, let $p = 1/(|a| + |b| + 1)$; then $1 - pa$ and $1 - pb$ will be positive, so that

$$f(1 - pa, 1 - pb) = \frac{1}{2}(1 - pa + 1 - pb) = 1 - p(a+b)/2.$$

But $f(1 - pa, 1 - pb) = 1 - f(pa, pb)$, by A2; $= 1 - pf(a, b)$, by III or III'. Therefore $f(a, b) = (a+b)/2$, for all real values of a and b . Hence Set A2 and Set A5 reduce to Set A1.

THEOREM A (C'). *Proof of IV from A3, I, III. (Real domain.)*

By III, $f(0, 0, \dots, 0) = 2f(0, 0, \dots, 0)$; hence $f(0, 0, \dots, 0) = 0$.

From A3, $f(1, 1, \dots, 1) = 1 - f(0, 0, \dots, 0) = 1$. Hence by III,

$$f(a, a, \dots, a) = a,$$

whenever a is positive.

From A3, $f(2, 2, \dots, 2) = 1 - f(-1, -1, \dots, -1)$, whence

$$2 = 1 - f(-1, -1, \dots, -1),$$

whence $f(-1, -1, \dots, -1) = -1$. Hence by III,

$$f(a, a, \dots, a) = a$$

whenever a is negative.

Since A2 follows immediately from A3, this proof shows that Set A3 reduces to Set A2.

THEOREM A (D'). *Proof of A1 and IV from A4, I, II. (Real domain.)*

If a and b are positive, put $A = a + b$, $x_1 = a$, $x_2 = b$, and $n = 2$, in A4; then $f(b, a) = a + b - f(a, b)$, whence by I, $f(a, b) = a + b - f(a, b)$, whence

$$f(a, b) = \frac{1}{2}(a+b),$$

whenever a and b are positive.

If a and b are any reals, let $p =$ a large positive quantity, so that $p+a$, $p+b$, and $p+a+b$ will certainly be positive. Putting $A = p+a+b$, $x_1 = a$, $x_2 = b$, and $n = 2$, in A4, we have

$$f(p+b, p+a) = p + a + b - f(a, b),$$

whence

$$(p + b + p + a)/2 = p + a + b - f(a, b),$$

whence $f(a, b) = (a + b)/2$, which is Postulate A1 for all real values of a and b .

If a is positive, put $x_1 = a, x_2 = a, \dots, x_n = a$, and $A = 2a$, in A4; then $f(a, a, \dots, a) = 2a - f(a, a, \dots, a)$, whence $f(a, a, \dots, a) = a$.

Again, if a is positive, put $x_1 = 0, x_2 = 0, \dots, x_n = 0, A = a$, in A4; then $f(a, a, \dots, a) = a - f(0, 0, \dots, 0)$, whence $f(0, 0, \dots, 0) = 0$.

If a is negative, put $x_1 = a, x_2 = a, \dots, x_n = a$, and $A = 0$, in A4; then $f(-a, -a, \dots, -a) = -f(a, a, \dots, a)$, whence, since $-a$ is positive, $-a = -f(a, a, \dots, a)$; hence $f(a, a, \dots, a) = a$.

Therefore $f(a, a, \dots, a) = a$ for all real values of a , which is Postulate IV. This proof shows that Set A4 reduces to Set A1.

THEOREM A (E'). Since IV follows at once from III, IV', IV'', we see that Set A6'' and Set A7'' reduce to Sets A2 and A1 respectively, in the domain of all reals.

THEOREM A (F'). *Proof of A4 from A8, A8'. (Real domain.)*

By A8, $f(A - x_1, A - x_2, \dots, A - x_n) = A + f(-x_1, -x_2, \dots, -x_n)$, whence, by A8', $f(A - x_1, A - x_2, \dots, A - x_n) = A - f(x_1, x_2, \dots, x_n)$. This shows that Set A8 reduces to Set A4, and hence determines the arithmetic mean in the domain of reals. Unlike the other sets, however, this Set A8 cannot be used if the n quantities $x_1, x_2, \dots, x_n$ are restricted to the domain of positive values.

APPENDIX II. POSTULATES FOR THE ARITHMETIC MEAN IN THE COMPLEX DOMAIN

In the domain of complex quantities, Sets A1 and A8 are sufficient to determine the arithmetic mean. (See proofs below.) Further, if we form a new postulate

POSTULATE A9. $f(A - x_1, A - x_2, \dots, A - x_n) = A - f(x_1, x_2, \dots, x_n)$, then a Set A9, comprising Postulates I, II, A9, will also be sufficient, as proved below. (This postulate A9 is the same as A4 without the restriction $x_i < 1$.)

Hence, in the complex domain, we have three sets of independent postulates for the arithmetic mean, as follows:

Set	Postulates				Examples Used			
A1	A1	I	II	IV	0	A I	A II''	A IV
A8	A8	I	II	A8'	0	A I	A II''	0'
A9	A9	I	II		0	A I	A II''	

The new example here required is

Ex. A II''. $f(x_1 + iy_1, x_2 + iy_2, \dots, x_n + iy_n) = x + iy$, where x is the median of the x 's if n is odd (or the arithmetic mean of their median-pair if n is even), and y is the median of the y 's if n is odd (or the arithmetic mean of their median-pair if n is even).

This example satisfies Postulates A1, A8, A8', A9, I and IV, but fails on Postulate II.

The properties of all these examples for the complex domain are shown in the following table:

	A1	A8	A8'	A9	I	II	IV
Ex. 0	A1	—	A8'	—	I	II	IV
Ex. 0'	A1	A8	—	—	I	II	IV
Ex. A I	A1	A8	A8'	A9	—	II	IV
Ex. A II''	A1	A8	A8'	A9	I	—	IV
Ex. A IV	A1	—	A8'	—	I	II	—

On the other hand, it is interesting to note that if the n quantities $x_1, x_2, \dots, x_n$ are allowed to take on complex values, then Sets A2, A3, A4, A5, A6, A7, A6'', A7'' are not sufficient to determine the arithmetic mean, as is shown by the following examples:

Ex. A IX.

$f(x_1 + iy, x_2 + iy, \dots, x_n + iy) = (x_1 + iy + x_2 + iy + \dots + x_n + iy)/n$ when all the y 's are equal; otherwise,

$$f(x_1 + iy_1, x_2 + iy_2, \dots, x_n + iy_n) = (x_1 + x_2 + \dots + x_n)/n.$$

This example satisfies Postulates I, II, III, IV, III', IV', IV'', V, A2, A3, A4, but it is not the arithmetic mean. (This example would still satisfy A2 and A3, even if we removed the restrictions $a < 1$, $b < 1$, and $x_i < 1$.)

Ex. A X. When $n=2$, $f(x_1, x_2) = (x_1 + x_2)/2$; when $n > 2$,

$$f() = (x_1 + x_2 + \dots + x_n)/n \text{ or } |(x_1 + x_2 + \dots + x_n)/n|,$$

according as $(x_1 + x_2 + \dots + x_n)/n$ is real or imaginary.

This example satisfies Postulates A1, I, II, III, IV', IV'', V, but fails on Postulate IV.

The proofs of the theorems in the complex domain, on account of their great simplicity, are here given in full, as follows:

THEOREM A (A''). *Proof of A from A1, I, II, IV. (Complex domain.)*

By II and A1, $f(x_1, x_2, \dots, x_n) = f(0, x_1 + x_2, x_3, \dots, x_n)$, since each side equals $f\{(x_1 + x_2)/2, (x_1 + x_2)/2, x_3, \dots, x_n\}$.

By successive applications of this result,

$$f(x_1, x_2, \dots, x_n) = f(0, 0, \dots, [x_1 + x_2 + \dots + x_n]).$$

Now let $a = (x_1 + x_2 + \dots + x_n)/n$, and take each x equal to a . Then $f(a, a, \dots, a) = f(0, 0, \dots, 0, na) = a$, by IV.

But also $f(x_1, x_2, \dots, x_n) = f(0, 0, \dots, 0, na)$.

Hence $f(x_1, x_2, \dots, x_n) = (x_1 + x_2 + \dots + x_n)/n$.

This proof shows that Set A1 determines the arithmetic mean, for all complex values of the variables.

THEOREM A (F''). *Proof of A1 and IV from A8, A8', I, II. (Complex domain.)*

Putting each $x=0$ in A8', we have $f(0, 0, \dots, 0)=0$; hence putting each $x=0$ and $A=a$, in A8, we have $f(a, a, \dots, a)=a$, which is Postulate IV.

Again, putting $A=a+b$, $x_1=-a$, $x_2=-b$, and $n=2$, in A8, we have $f(b, a) = a+b+f(-a, -b)$, whence, by I and A8', $f(a, b) = a+b-f(a, b)$, whence $f(a, b) = (a+b)/2$, which is Postulate A1.

This proof shows that Set A8 reduces to Set A1, for all complex values.

THEOREM A (G). *Proof of A1 and IV from A9, I, II. (Complex domain.)*

In A9, put each x equal to 0, and $A=0$; then

$$f(0, 0, \dots, 0) = -f(0, 0, \dots, 0),$$

whence $f(0, 0, \dots, 0)=0$.

Hence, putting each x equal to a , and $A=a$, we have

$$f(0, 0, \dots, 0) = a - f(a, a, \dots, a),$$

whence $f(a, a, \dots, a) = a$, which is Postulate IV for all values of a .

Again, put $A=a+b$, $x_1=a$, $x_2=b$, and $n=2$, in A9; then

$$f(b, a) = a + b - f(a, b),$$

whence, by I, $f(a, b) = a + b - f(a, b)$, whence $f(a, b) = (a+b)/2$, which is Postulate A1 for all values of a .

This proof shows that Set A9 reduces to Set A1, for all complex values.

The sufficiency of each of the Sets A1, A8, A9, is thus established.

In conclusion, it may be noted that of all the known sets of postulates for the arithmetic mean, the only ones that are equally available in the positive domain, the real domain, and the complex domain, are Sets A1 and A9. Of these, Set A9 (consisting of Postulates A9, I, II) would appear to be the simplest.

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